"$f \boxslash g$" means "lift exists in any
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$
"

- $\mathcal{A}^{\boxslash} = \{ g \mid a \boxslash g \ \forall\ a \in \mathcal{A} \}$ right complement
- ${}^{\boxslash}\mathcal{B} = \{ f \mid f \boxslash b \ \forall\ b \in \mathcal{B} \}$ left complement

$\overline{\mathcal{A}} \subseteq {}^{\boxslash}(\mathcal{A}^{\boxslash})$ is weakly saturated

- was $\supseteq \mathcal{A}$
- closed under:
  - cobase change
  - transfinite composition
  - retracts

($\Rightarrow$ • composition, coproducts)

$\text{InnHorn} = \{ \Lambda^n_k \subset \Delta^n,\ 0<k<n \}$

$\Rightarrow$ $\overline{\text{InnHorn}}$ $\quad$ $\text{InnFib} := \text{InnHorn}^{\boxslash}$

Prop: If S is a set of maps in sSet, then any map f in sSet admits $f = pj$ where $j \in \overline{S}$, $p \in S^{\boxslash}$

Cor: S set of maps $\Rightarrow$ $\overline{S} = {}^{\boxslash}(S^{\boxslash})$

Proof: $\overline{S} \supseteq {}^{\boxslash}(S^{\boxslash})$

Suppose $f \boxslash (S^{\boxslash})$, By Prop: $f = pj$, $j \in \overline{S}$, $p \in S^{\boxslash}$

$\Downarrow$

$$\begin{tikzcd} \cdot \arrow[r, "id"] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & \cdot \arrow[r, "id"] \arrow[d, "j"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, dashed, "s"', "\exists ?"] \arrow[rr, bend right, "id"'] & \cdot \arrow[r, "p"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "j"] \arrow[d, "f"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "id"'] \arrow[ur, dashed, "\exists"] & \cdot \end{tikzcd}$$

$\Rightarrow$ f is a retract of $j \in \overline{S}$

$\Rightarrow$ $f \in \overline{S}$.

inner anodyne → $\overline{\text{InnHorn}} = {}^{\boxslash}\text{InnFib}$

$\overline{\text{InnHorn}}^{\boxslash} = \text{InnFib}$ $\Rightarrow$ any $f = pj$, $j \in \overline{\text{InnHorn}}$, $p \in \text{InnFib}$

Weak factorization system: pair $(\mathcal{L}, \mathcal{R})$ of classes of maps st

(1) every f admits $f = pj$, $j \in \mathcal{L}$, $p \in \mathcal{R}$

(2) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$, $\mathcal{R} = \mathcal{L}^{\boxslash}$ $\Rightarrow$ "$\mathcal{L} \boxslash \mathcal{R}$"

e.g. $(\overline{\text{InnHorn}}, \text{InnFib})$
inner anodyne, inner fibrations

Small-object argument:

$S = \{ s_i : A_i \to B_i \}$ set of maps

Want: $f = pj$, $j \in \overline{S}$, $p \in S^{\boxslash}$

Given $f : X \to Y$ $\Rightarrow$

$$\begin{tikzcd} X \arrow[r, "Lf"] \arrow[rr, bend right, "f"'] & Ef \arrow[r, "Rf"] & Y \end{tikzcd}$$

by $[S, f] := \{ (s_i, u, v) \mid s_i \in S, fu = v s_i \}$

$$= \left\{ \begin{tikzcd} A_i \arrow[r, "u"] \arrow[d, "s_i"'] & X \arrow[d, "f"] \\ B_i \arrow[r, "v"'] & Y \end{tikzcd} \right\} \text{ set}$$

$$\begin{tikzcd} \coprod_{(s_i,u,v)} A_i \arrow[r, "(u)"] \arrow[d, "(s_i)"'] & X \arrow[d, "Lf"] \arrow[ddr, bend left, "f"] & \\ \coprod_{(s_i,u,v)} B_i \arrow[r] \arrow[dr, bend right, "v"'] & Ef \arrow[dr, "Rf"] \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & Y \end{tikzcd}$$

$Lf :=$ pushout $\Rightarrow Lf \in \overline{S}$.

$Rf \notin S^{\boxslash}$ (but almost)

$$\begin{tikzcd}
 & & X \arrow[d, "Lf"] \\
A_i \arrow[urr, bend left, "\tilde{u}"] \arrow[rr, "u"] \arrow[d, "S_i"'] & & Ef \arrow[d, "Rf"] \\
B_i \arrow[rr, "v"'] \arrow[urr, dashed] & & Y
\end{tikzcd}$$

$\exists$ lift exists by construction if u factors through Lf

in $\bar{S}$

Iterate

$$\begin{tikzcd}
X \arrow[d, equal] \arrow[dr, bend left, "Lf"] \arrow[drr, bend left] \arrow[drrr, bend left] \arrow[drrrr, bend left] \arrow[drrrrrr, bend left, "L^\omega f"] & & & & & & \\
X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[r, "LEf"] \arrow[dl, "Rf"] & E^2 f \arrow[r] \arrow[dll, "REf"] & E^3 f \arrow[r] \arrow[dlll] & E^4 f \arrow[r] \arrow[dllll] & \cdots \arrow[r] & E^\omega f = \operatorname{colim}_{\alpha<\omega} E^\alpha f \arrow[dllllll, "R^\omega f"] \\
Y & & & & & &
\end{tikzcd}$$

$\bar{S}$

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, bend left, "\bar{S}"] & \\
E^\omega f \arrow[r] \arrow[d] & E^{\omega+1} f \arrow[dl] \\
Y &
\end{tikzcd}$$

$\Rightarrow$ for any ordinal α we get

$$\begin{tikzcd}
X \arrow[d, "L^\alpha f"'] & \bar{S} \arrow[l] \\
E^\alpha f \arrow[d, "R^\alpha f"'] & \\
Y &
\end{tikzcd}$$

<u>Claim</u>: $\exists$ ordinal κ such that for any $S_i : A_i \to B_i \in \underline{S}$

for any $A_i \xrightarrow{u} E^\kappa f$, $\exists$ $\alpha < \kappa$.

st u factors through $E^\alpha f$:

$$\begin{tikzcd}
 & E^\alpha f \arrow[d] \\
A_i \arrow[ur, "\exists \hat{u}"] \arrow[d] & E^{\alpha+1} f \arrow[d] \\
B_i \arrow[ur, dashed] \arrow[ddr, bend right, "v"'] & E^\kappa f \arrow[d] \\
 & \\
 & Y
\end{tikzcd}$$

Given this, $X \xrightarrow{L^\kappa f} E^\kappa f \xrightarrow{R^\kappa f} Y$

$L^\kappa f \in \bar{S}$, $R^\kappa f \in S^\boxslash$

$\kappa :=$ sufficiently regular cardinal e.g. larger than # of simplices in A_i

Regular cardinal : κ : infinite cardinal s.t.

for every set A of ordinals with

(1) $\alpha < \kappa, \quad \forall \alpha \in A$

(2) $|A| < \kappa$

$\Rightarrow \quad \sup A < \kappa$

$\Rightarrow$ K - simplicial set with $< \kappa$ cells

$$\operatorname{Hom}(K, \operatorname{colim}_{\lambda<\kappa} X_\lambda) = \operatorname{colim}_{\lambda<\kappa} \operatorname{Hom}(K, X_\lambda)$$

ie, every $K \to \operatorname{colim}_{\lambda<\kappa} X_\lambda$ factors through some $X_\lambda \to$ colim

(if $|K| < \kappa$)

$(\text{InnHorn}, \text{InnHorn}^{\boxslash})$

want W.F.S with $(\text{monomorphism}, \sim)$

$\text{monomorphism} \overset{\shortparallel}{} \text{Cell}$

Boundary of standard n-simplex: $\Delta^n = \operatorname{Hom}_\Delta(-, [n])$

Subcomplex $\partial\Delta^n \subseteq \Delta^n$

$$\partial\Delta^n := \bigcup_{k \in [n]} \Delta^{[n] \setminus k} \subsetneq \Delta^n$$

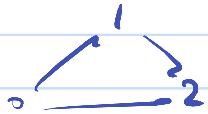
$$(\partial\Delta^n)_j = \{ f : [j] \to [n] \mid f([j]) \neq [n] \}$$

$\partial\Delta^0 = \emptyset$ $\qquad\qquad$ $\partial\Delta^1$ $\quad 0 \qquad 1$

$\cap$

Δ^0 $\quad \bullet$ $\qquad\qquad$ Δ^1 $\quad 0 \longrightarrow 1$

$\partial\Delta^2$



Δ^2

Rem: C category $\implies$ $\mathrm{Hom}(\Delta^n, C) \longrightarrow \mathrm{Hom}(\partial\Delta^n, C)$ bijection if $n \geq 3$

$\mathrm{Cell} := \{ \partial\Delta^n \subset \Delta^n, \ n \geq 0 \}$ ← monomorphisms

$\implies$ $\overline{\mathrm{Cell}}$ weak saturation $\subseteq$ monomorphisms

$\mathrm{Cell}^{\boxslash} =: \mathrm{TrivFib}$ trivial fibrations

$\implies$ $(\overline{\mathrm{Cell}}, \mathrm{Cell}^{\boxslash})$ W.F.S.

Goal: show $\overline{\mathrm{Cell}} = \text{monomorphisms}$

$\Delta \supset \Delta^{surj}, \Delta^{inj}$ subcategories of surjective/injective simplicial operations

every $f : [m] \longrightarrow [n]$ in Δ factors uniquely

as $f = f^{inj} f^{surj}$ $\qquad$ $f^{surj} \in \Delta^{surj}$

$f^{inj} \in \Delta^{inj}$

$X \in sSet$

$a \in X_n$ is **degenerate** if $\exists\ f \in \Delta \setminus \Delta^{inj}$, $b \in X$. st $a = bf$

(equiv, $\exists\ f \in \Delta^{surj}$, $f \neq id$, $b \in X$ st. $a = bf$)

$X_n^{deg} \subseteq X_n$ set of degenerate n-cells

$$X_n^{deg} = \bigcup_{\substack{f:[n]\to[m]\\ n>m}} X_m f$$

$a \in X_n$ is non-degenerate if it is not degenerate

ie, if $a = bf$, then $f \in \Delta^{inj}$

($\Leftrightarrow$ if $a = bf$, $f \in \Delta^{surj}$ $\Rightarrow$ $f = id$.)

$$X_n = X_n^{deg} \amalg X_n^{nd}$$

if $f : X \to Y$, $f(X_n^{deg}) \subseteq Y_n^{deg}$

$f^{-1}(Y_n^{nd}) \subseteq X_n^{nd}$

Prop: $A \subseteq X$, $A_n^{nd} = X_n^{nd} \cap A_n$

$A_n^{deg} = X_n^{deg} \cap A_n$

Pf: $A_n^{deg} \subseteq X_n^{deg} \cap A_n$ ✓

$a \in X_n^{deg} \cap A_n$ so $a = bg$, $g : [n] \to [k] \in \Delta^{surj}$, $b \in X_k$, $n > k$

Fact: g has a section $s : [k] \to [n]$, $gs = id_{[k]}$

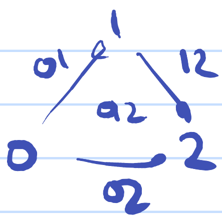
$b = bgs = as \in A_k$

Ex: $f \in (\Delta^n)_k$, $\qquad f : [k] \to [n]$

degenerate iff f non-injective

non-degenerate iff f injective

Δ^2



$00,\ 11,\ 22 \in (\Delta^2)_1^{deg}$

$000,\ 111,\ 222,$
$001,\ 011,\ 002,$
$022,\ 112,\ 122 \quad \in (\Delta^2)_2^{deg}$

Ex: $\Delta^n / \partial\Delta^n = \mathrm{colim}\left(* \leftarrow \partial\Delta^n \hookrightarrow \Delta^n \right)$

$$(\Delta^n/\partial\Delta^n)_k = (*)_k \sqcup (\Delta^n)_k \setminus (\partial\Delta^n)_k$$

has only two non deg cells in dim $0, n$

$*_0, \quad \bar{\iota} \in \dim n$

(image of $1 \in (\Delta^n)_n$)

Prop: (Eilenberg-Zilber): $\quad a \in X$

then $\exists$ unique (b, σ), $\quad b \in X^{nd}$, $\sigma \in \Delta^{surj}$

st $\quad a = b\sigma$

Cor:
$$\coprod_{j \geq 0} X_j^{nd} \times \mathrm{Hom}_{\Delta^{surj}}([n],[j]) \xrightarrow[\cong]{\text{by}} X_n$$
$$(b, \sigma) \longmapsto b\sigma$$

"$X : \Delta^{op} \to Set$, $\quad X|_{(\Delta^{surj})^{op}} : (\Delta^{surj})^{op} \to Sets$

is canonically free"

Pf. EZ: a dg $\Rightarrow$ $a = b\sigma$ $\quad \sigma \in \Delta^{surj}$

choose b to minimal dim $\Rightarrow$ $b \in X^{nd}$

a nd, $\quad a = a \cdot id$

Given $\sigma : [n] \to [m]$, $\quad \Gamma(\sigma) = \{ \underset{\text{section}}{\delta : [m] \to [n]},\ \sigma\delta = id_{[m]} \}$

$\sigma \in \Delta^{surj} \iff \Gamma(\sigma) \neq \emptyset$

(Fact: $\underset{[n] \to [n]}{\sigma, \sigma' \in \Delta^{surj}}$: $\quad \Gamma(\sigma) = \Gamma(\sigma') \Rightarrow \sigma = \sigma'$)

$a \in X_n$, $\quad a = b_1\sigma_1 = b_2\sigma_2$, $\quad b_j \in X^{nd}_{m_j}$, $\sigma_j : [n] \twoheadrightarrow [m_j]$

WTS: $\quad m_1 = m_2$, $b_1 = b_2$, $\sigma_1 = \sigma_2$

Pick: $\quad \delta_1 \in \Gamma(\sigma_1)$, $\delta_2 \in \Gamma(\sigma_2)$

$b_1 = b_1\sigma_1\delta_1 = a\delta_1 = b_2(\sigma_2\delta_1)$

$b_2 = b_2\sigma_2\delta_2 = a\delta_2 = b_1(\sigma_1\delta_2)$

$\delta \quad \sigma \quad id = \sigma\delta$

$b_1, b_2 \in X^{nd}$, $\quad \Rightarrow \quad \sigma_2\delta_1 : [m_1] \to [m_2]$ $\quad$ injective $\quad m_1 \le m_2$

$\sigma_1\delta_2 : [m_2] \to [m_1]$ $\quad m_2 \le m_1$

$\Rightarrow$ $m = m_1 = m_2$ $\quad \sigma_2\delta : [m] \rightarrowtail [m]$ $\quad \Rightarrow \quad \sigma_2\delta_1 = id_{[m]} = \sigma_1\delta_2$

$\Rightarrow \quad \Gamma(\sigma_1) = \Gamma(\sigma_2) \quad \Rightarrow \quad \sigma_1 = \sigma_2$ ✓

Cell = monomorphisms $\leftarrow$ skeletal filtration of a s.set

If X is s.t. if $a \in X^{nd}$, $\delta \in \Delta^{inj}$, $\Rightarrow a\delta \in X^{nd}$

(<u>YES</u>: Δ^n, <u>NO</u>, $\Delta^n/\partial\Delta^n$ if $n>2$)

combinatorics of a

$\Rightarrow X \longleftrightarrow$ "<u>Δ-complex</u>" in Hatcher.

↑ only need nondeg cells + face relations to recover X.